# Relative Magnetic Helicity Based on a Periodic Potential Field

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## Abstract

Magnetic helicity is conserved under ideal magnetohydrodynamics and quasi-conserved even under a resistive process. The standard definition for magnetic helicity cannot be applied directly to an open magnetic field in a volume, because it is gauge-dependent. Instead, the relative magnetic helicity is widely used. We find that the energy of a potential magnetic field in a rectangular domain with periodic lateral boundary conditions is less than that of the field with a fixed normal component on all six boundaries. To make use of this lower energy potential field in the analysis of relative magnetic helicity, we introduce a new definition for magnetic helicity for the magnetic field, which involves the periodic potential field. We apply this definition to a sequence of analytic solutions and a numerical simulation. The results show that our new gauge-invariant helicity is very close to the current-carrying part of the relative magnetic helicity of the original magnetic field. We find also that the ratio between the current-carrying helicity and the relative magnetic helicity for the original and our defined relative helicity show different behavior. It seems that the new helicity is more sensitive to the component of the field due to the electric current in the volume, which is the source for instabilities and solar eruptive phenomena.

## Model

**Old:** In the classical definition, a magnetic field $\mathbf{B}$ in a three-dimension (3D) volume, $\Omega$, will be decomposed as $\mathbf{B}_j+\mathbf{B}_p$, with the boundary condition $(\mathbf{B}-\mathbf{B}_p)\cdot\hat{\mathbf{n}}|_{\partial\Omega}=0$, where $\partial\Omega$ is the boundary and $\hat{\mathbf{n}}$ is the associated unit normal vector. Thus $\mathbf{B}_p$ can play the role of the reference field. With this decomposition, Finn & Antonsen (1985) define the relative magnetic helicity as,

$$H_r = \int_\Omega (\mathbf{A}+\mathbf{A}_p)\cdot(\mathbf{B}-\mathbf{B}_p)\mathrm{d}^3\mathbf{x}. \quad (1)$$

This formula is widely used in both theoretical and numerical computation. Specifically, Berger (1999) show that this relative helicity can be rewritten as the sum of another two gauge-invariant helicity $H_r = H_j + H_{pj}$,

$$H_j = \int_\Omega (\mathbf{A}-\mathbf{A}_p)\cdot(\mathbf{B}-\mathbf{B}_p)\mathrm{d}^3\mathbf{x}, \quad (2)$$

and

$$H_{pj} = 2\int_\Omega \mathbf{A}_p\cdot(\mathbf{B}-\mathbf{B}_p)\mathrm{d}^3\mathbf{x}. \quad (3)$$

**New:** Consider a cartesian domain, we can decompose a 3D magnetic field $\mathbf{B}$ into a current-associated field and a potential field with periodic boundary condition, $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_c$. In this case $\mathbf{B}_0$ does not match the lateral boundary condition on $\mathbf{B}$, and therefore, $\mathbf{B}_c\cdot\hat{\mathbf{n}}$ on the lateral boundaries does not vanish. It is not possible to use the periodic potential field, $\mathbf{B}_0$, as a reference field for the relative magnetic helicity, required by the gauge-invariant. Following the original definition from Berger & Field (1984), we can decompose $\mathbf{B}_c$ into two parts, $\mathbf{B}_c=\mathbf{B}_{c1}+\mathbf{B}_{p1}$, where $\mathbf{B}_{p1}$ is the solution of Laplace's equation that satisfies the boundary condition $(\mathbf{B}_c-\mathbf{B}_{p1})\cdot\hat{\mathbf{n}}|_{\partial\Omega}=0$ on all boundaries. Considering Finn & Antonsen (1985), we can then define a gauge-invariant relative magnetic helicity for the field $\mathbf{B}_c$,

$$H_{cr} = \int_\Omega (\mathbf{A}_c+\mathbf{A}_{p1})\cdot(\mathbf{B}_c-\mathbf{B}_{p1})\mathrm{d}^3\mathbf{x}, \quad (4)$$

where, $\mathbf{A}_c$ and $\mathbf{A}_{p1}$ are the corresponding vector potential field. Obviously, this helicity can be decomposed into another two gauge-invariant helicity, $H_{cr}=H_{cj}+H_{cpj}$, where

$$H_{cj} = \int_\Omega (\mathbf{A}_c-\mathbf{A}_{p1})\cdot(\mathbf{B}_c-\mathbf{B}_{p1})\mathrm{d}^3\mathbf{x}, \quad (5)$$

and

$$H_{cpj} = 2\int_\Omega \mathbf{A}_{p1}\cdot(\mathbf{B}_c-\mathbf{B}_{p1})\mathrm{d}^3\mathbf{x}. \quad (6)$$

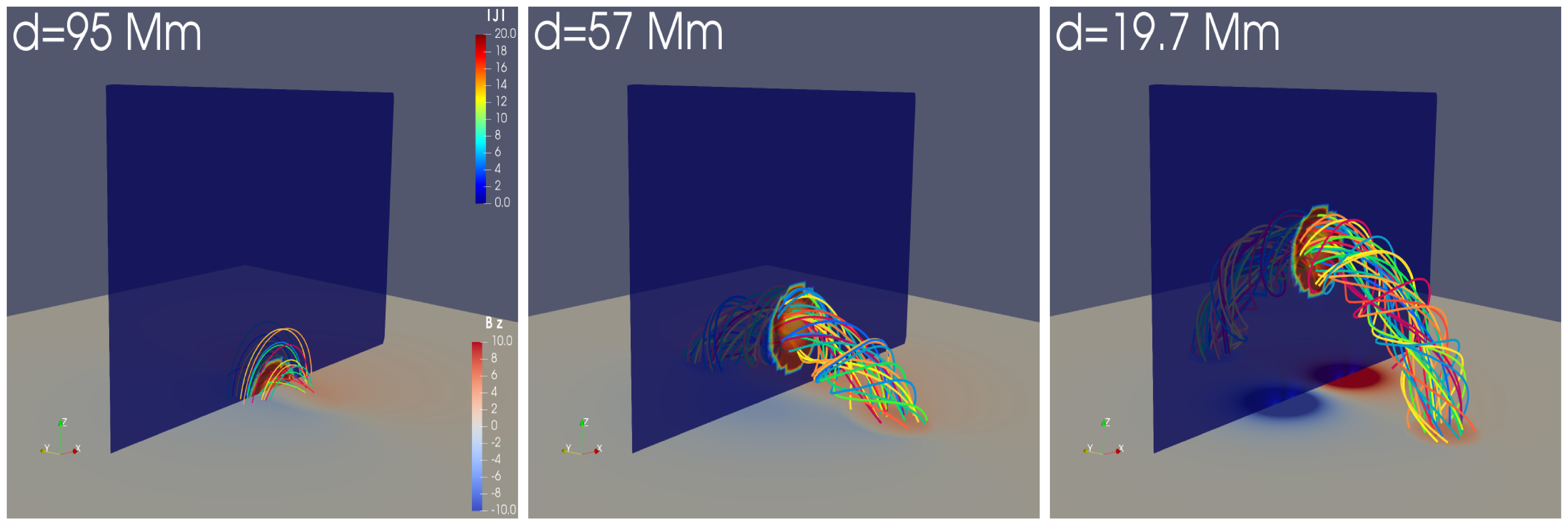
**Difference:** The most important difference between the original helicity and our definition is that our one, $H_{cr}$, is much closer to the mutual helicity between the current-carrying part of the field and the potential field, $H_{pj}$.

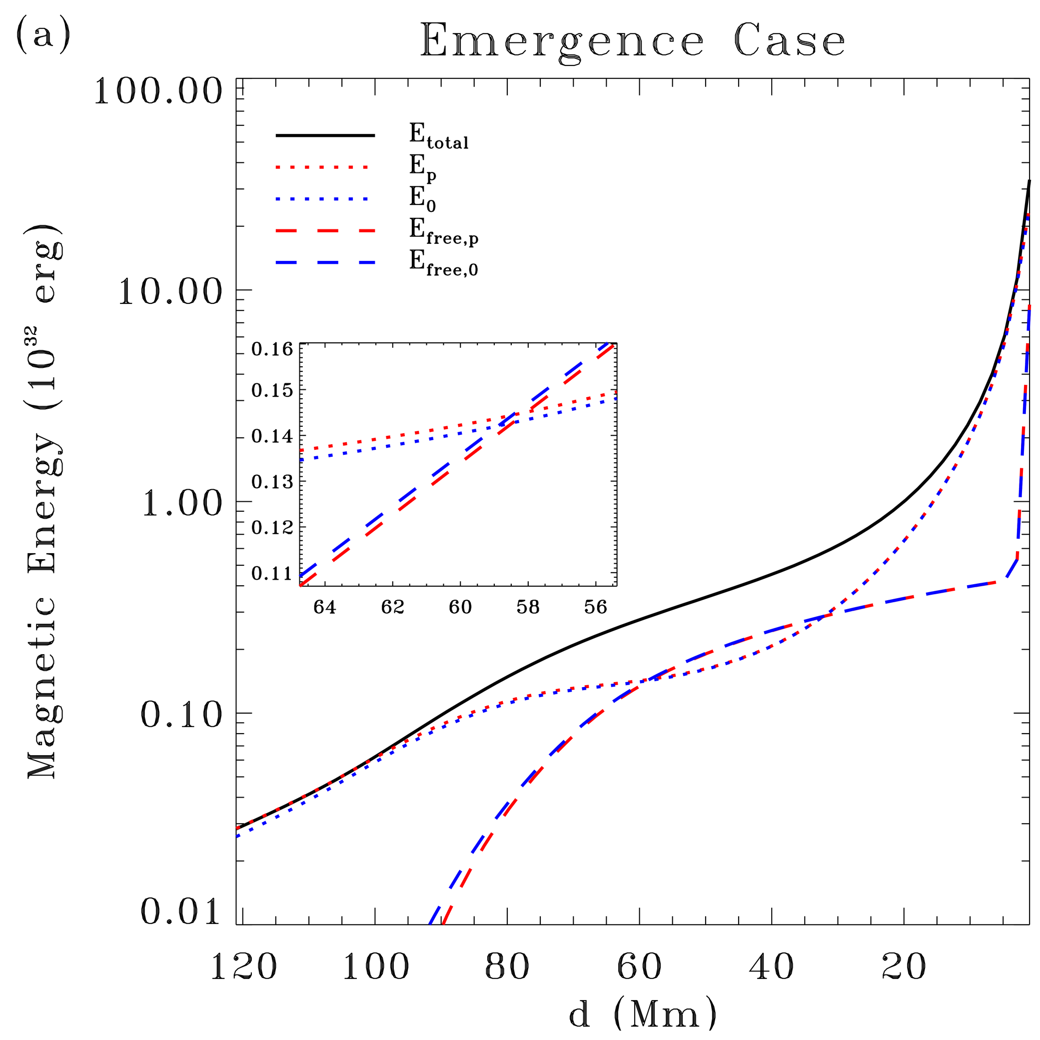
## Result

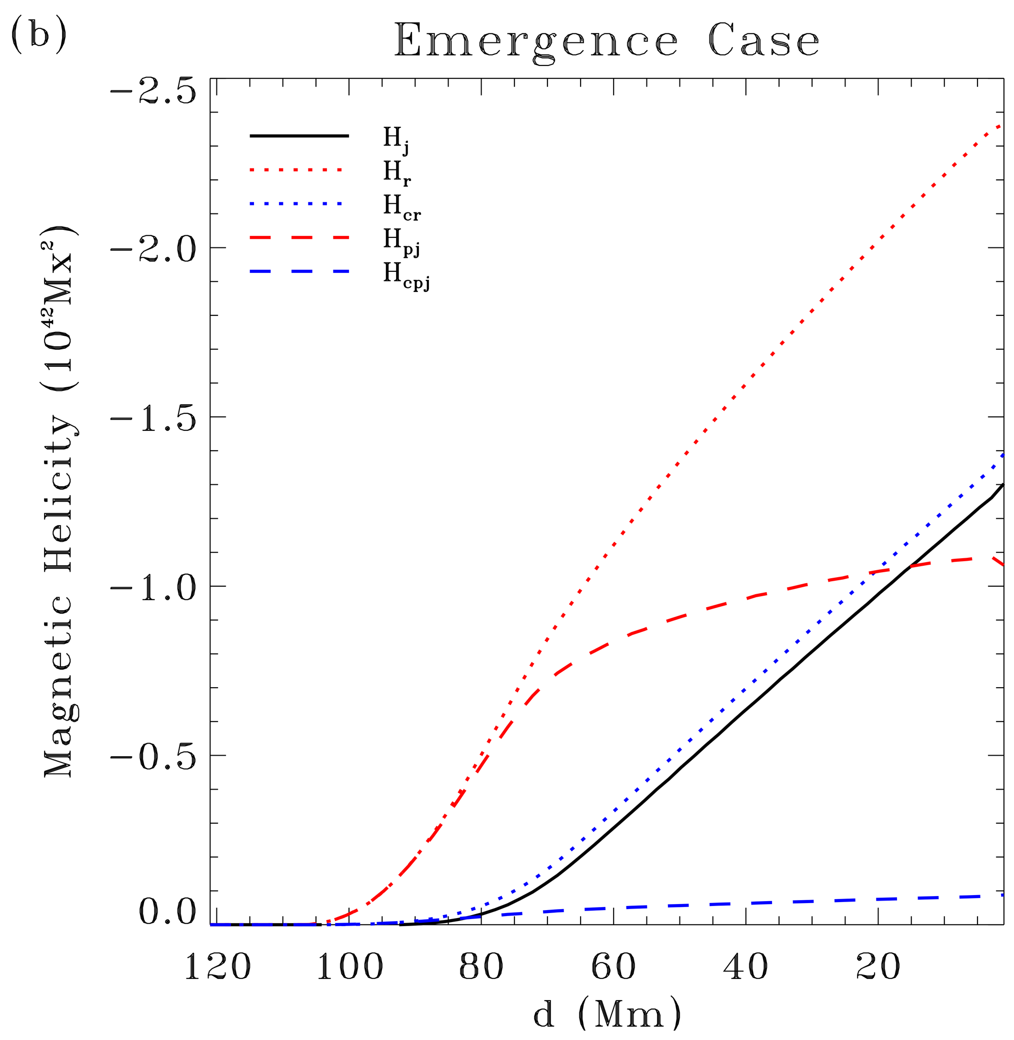
The model are applied to two two cases, a pseudo emergence processs and a eruptive case. We decrease the depth parameter in a series of Titov-Démoulin flux-rope models to mimic an artificial emergence process for the current system, helicity and energy are injected into the computational domain similar to what happens during the emergence of a solar active region. However, there is no physical flow on the boundary, so the associated injection flux can not be calculated directly. The eruptive case come a isothermal MHD simulation using TD model as the initial condition. In both cases, the potential field component, $\mathbf{B}_{p1}$ is very small, which makes $H_{cpj}$ close to zero because it is the coupling between this component of the potential field and the current-carrying part. Therefore, the value of $H_{cr}$ is very close to that of $H_j$ and shows a monotonic increase during the whole simulation period. In the eruptive case, the helicity associated with the potential field $\mathbf{B}_p$ does not show a departure from $H_r$ in this case. That is due to the boundary conditions: $B_z$ is fixed in this case; whereas in the emergence case, $B_z$ changes due to the line current and magnetic charges approaching the lower boundary.

## Tests

**Pseudo emergence case**




**Eruptive case**